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Some New Characterizations of the Chebyshev Polynomials

C. A. MICCHELLI AND T. J. RIVLIN

IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598

INTRODUCTION

Let P_n denote the set of polynomials of degree at most n, and let Q_n be the subset of P_n consisting of $p \in P_n$ which satisfy

$$|p(\eta_j)| = 1, \qquad j = 0,..., n, \tag{1}$$

where

$$\eta_j = \cos(j\pi/n), \qquad j = 0,..., n.$$

We conjectured [3] that if $p \in Q_n$ has n simple zeros in $[-1, 1]$ then $p = \pm T_n$, $T_n(x)$ being the Chebyshev polynomial of degree n. This conjecture was recently proved by DeVore [1]. Our purpose here is to present a different and self-contained proof of a generalization of our conjecture, and to give another characterization of the Chebyshev polynomials which complements that of DeVore. We shall also give the application to the theory of numerical integration which suggested the original conjecure.

1. TWO CHARACTERIZATIONS OF CHEBYSHEV POLYNOMIALS

Our generalization of DeVore [1] is Theorem 1.

THEOREM 1. *If $p \in Q_n$ has all its zeros in the (lemniscatic) set of the complex plane*

$$S = \{z / |1 - T_n^2(z)| \leqslant 1\},$$

then $p = \pm T_n$.

Proof. Let

$$p(z) = a(z - z_1) \cdots (z - z_n).$$

Then,

$$
\begin{aligned}
1 &= |\, p(\eta_0) \cdots p(\eta_n)| \cdot |\, p(\eta_1) \cdots p(\eta_{n-1})| \\
&= |\, a\,|^{n+1} \prod_{j=1}^{n} \left| \prod_{i=0}^{n} (\eta_i - z_j) \right| \cdot |\, a\,|^{n-1} \prod_{j=1}^{n} \left| \prod_{i=1}^{n-1} (\eta_i - z_j) \right| \\
&= \frac{|\, a\,|^{2n}}{[n^2 2^{2(n-1)}]^n} \prod_{j=1}^{n} |(1 - z_j^2)\, [T_n'(z_j)]^2\, | \\
&= \frac{|\, a\,|^{2n}}{2^{2n(n-1)}} \prod_{j=1}^{n} |\, 1 - T_n^2(z_j)| \leqslant \frac{|\, a\,|^{2n}}{2^{2n(n-1)}},
\end{aligned}
$$

where we have used the identity $(1 - z^2)[T_n'(z)]^2 = n^2(1 - T_n^2(z))$. Thus we have

$$|\, a\,| \geqslant 2^{n-1}.$$

However, considering the divided difference of p at the η_j yields

$$a = p(\eta_0, \ldots, \eta_n) = \frac{2^{n-1}}{n} \sum_{j=0}^{n}{}'' (-1)^j\, p(\eta_j),$$

(the double prime on the sum means that the terms corresponding to $j = 0$ and $j = n$ are to be halved) and so $|\, a\,| \leqslant 2^{n-1}$, with equality possible if, and only if $p(\eta_j) = \pm(-1)^j$, $j = 0, \ldots, n$, that is, if, and only if, $p = \pm T_n$.

Remark. The interval $[-1, 1]$ is a subset of S.

Note that the condition (1) can be written as

$$|\, p(\eta_j)| = |\, T_n(\eta_j)|, \qquad j = 0, \ldots, n.$$

This suggests the following companion piece to Theorem 1.

THEOREM 2. *If $p \in P_n$ satisfies*

$$|\, p'(\xi_j)| = |\, T_n'(\xi_j)|, \qquad j = 1, \ldots, n, \tag{2}$$

where

$$\xi_j = \cos \frac{(2j - 1)\pi}{2n}, \; j = 1, \ldots, n,$$

and p' has all its zeros in the set of the complex plane

$$V = \{z / |\, T_n(z)| \leqslant 1\},$$

then $p' = \pm T_n'$.

Proof. Since $|T_n'(\xi_j)| = n/(1-\xi_j^2)^{1/2}$ and

$$\left[\prod_{j=1}^{n}(1-\xi_j^2)\right]^{-1/2} = \left[\frac{T_n(1)}{2^{n-1}}\,\frac{|T_n(-1)|}{2^{n-1}}\right]^{-1/2} = 2^{n-1},$$

(2) implies that if $p'(z) = c(z-z_1)\cdots(z-z_{n-1})$ then

$$2^{n-1}n^n = \left|\prod_{j=1}^{n}(p'(\xi_j)\right| = \left|\prod_{j=1}^{n} c \prod_{i=1}^{n-1}(\xi_j - z_i)\right|$$

$$= |c|^n \prod_{i=1}^{n-1}\frac{|T_n(z_i)|}{2^{n-1}} \leqslant \frac{|c|^n}{2^{(n-1)^2}}.$$

Hence, $|c| \geqslant n2^{n-1}$.

However,

$$c = p'(\xi_1,..., \xi_n) = 2^{n-1}\sum_{j=1}^{n}\frac{p'(\xi_j)}{T_n'(\xi_j)},$$

yielding $|c| \leqslant n2^{n-1}$, with equality possible if, and only if, $p'(\xi_j) = \epsilon T_n'(\xi_j)$ (where $\epsilon = \pm 1$), $j = 1,..., n$, that is, if, and only if, $p' = \pm T_n'$.

Remark. The interval $[-1, 1]$ is a subset of V.

2. An Application to Numerical Integration

In Micchelli and Rivlin [2] we gave numerical integration formulae of highest degree of precision for

$$A_n(f) = \frac{2}{\pi}\int_{-1}^{1} f(x)\, T_n(x)\,\frac{dx}{(1-x^2)^{1/2}},$$

namely,

$$A_n(f) = \frac{1}{n}\sum_{j=0}^{n}{}''(-1)^j f(\eta_j), \qquad f \in P_{3n-1}, \tag{3}$$

and

$$A_n(f) = \frac{1}{n}\sum_{j=1}^{n}\frac{f'(\xi_j)}{T_n'(\xi_j)}, \qquad f \in P_{3n-1}. \tag{4}$$

We observed in [2] that while there is no formula

$$A_n(f) = \sum_{i=1}^{n}[a_i f(x_i) + b_i f'(x_i)] \tag{5}$$

valid for $f \in P_{3n}$ (hence (4) is of highest degree of precision) we did not know (for $n \geqslant 4$) if (4) was the unique formula of the form (5), having all its nodes $[-1, 1]$, and valid for $f \in P_{3n-1}$. However, with Theorem 1 at our disposal we can establish the uniqueness of (4). To this end we use the following.

LEMMA. *There exist $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ such that* (5) *is valid for $f \in P_{3n-1}$ if, and only if, upon putting*

$$w(x) = (x - x_1) \cdots (x - x_n),$$

we have

$$(w(x)/w(1)) \in Q_n .$$

Proof. (5) holds for $f \in P_{3n-1}$ if, and only if,

$$\int_{-1}^{1} w^2(x)\, p(x)\, T_n(x) \frac{dx}{(1 - x^2)^{1/2}} = 0, \qquad p \in P_{n-1} . \tag{6}$$

Hence, in view of (3), (6) holds if, and only if,

$$\sum_{j=0}^{n}{}'' (-1)^j\, p(\eta_j)\, w^2(\eta_j) = 0, \qquad p \in P_{n-1} . \tag{7}$$

As a consequence of (3) and the orthogonality of the Chebyshev polynomials we have

$$\sum_{j=0}^{n}{}'' (-1)^j\, p(\eta_j) = 0, \qquad p \in P_{n-1} . \tag{8}$$

Subtracting (8) from (7) yields

$$\sum_{j=0}^{n-1}{}' p(\eta_j)\, [(-1)^j\, (w^2(\eta_j) - w^2(\eta_n))] = 0, \qquad p \in P_{n-1} \tag{9}$$

(the prime on the sum means that the term corresponding to $j = 0$ is to be halved) which is, therefore, equivalent to (6). Thus if $[w(x)/w(1)] \in Q_n$, (9) holds and there exists a formula (5) valid for $f \in P_{3n-1}$. Also if (9) holds, then if we take p in (9) to be $1, x, \ldots, x^{n-1}$, successively, we obtain a homogeneous system of n linear equations with nonzero determinant. Thus $w^2(\eta_j) = w^2(\eta_n)$, $j = 0, \ldots, n - 1$. Since $w^2(\eta_n) \neq 0$, for otherwise w has $n + 1$ zeros, we see that

$$(w(x)/w(1)) \in Q_n .$$

We now have

THEOREM 3. *The only numerical integration formula of the form* (5) *valid for* $f \in P_{3n-1}$ *with* $x_i \in S$, $i = 1,\dots, n$, *is* (4).

Proof. If (5) with $x_i \in S$, $i = 1,\dots, n$ is valid for $f \in P_{3n-1}$ then according to the lemma, $[w(x)/w(1)] \in Q_n$ and hence by Theorem 1, $x_i = \xi_i$, $i = 1,\dots, n$, (possibily after renumbering the x_i). For some fixed j, $1 \leqslant j \leqslant n$ let $g_j \in P_{2n-1}$ satisfy

$$g_j(\xi_i) = \delta_{ij}, \qquad i = 1,\dots, n,$$

$$g_j'(\xi_i) = 0, \qquad i = 1,\dots, n.$$

Putting $f = g_j$ in (4) yields $A_n(g_j) = o$, while substituting g_j for f in (5) now gives $a_j = 0$. Thus $a_j = 0$, $j = 1,\dots, n$, and as we showed in [2], $b_i = [nT_n'(\xi_i)]^{-1}$, $i = 1,\dots, n$.

We conclude with the observation that if we allow nodes outside of S, (4) need not be unique. For example, when $n = 2$, consider $w(x) = x^2 + x - 1$. Since $[w(x)/w(1)] \in Q_2$, its zeros $x_1 = -(1 + \sqrt{5})/2$ and $x_2 = (-1 + \sqrt{5})/2$ are the nodes of a quadrature formula (5) for $A_2(f)$ exact for $f \in P_5$, in view of the Lemma. However $x_1 \notin S = \{z/ | 4z^2(1 - z^2)| \leqslant 1\}$.

REFERENCES

1. R. A. DEVORE, A property of Chebyshev polynomials, *J. Approximation Theory* **12** (1974), 418–419.
2. C. A. MICCHELLI AND T. J. RIVLIN, Turan formulae and highest precision quadrature rules for Chebyshev coefficients, *IBM J. Res. Develop.* **16**, 1972, 372–379.
3. "Linear Operators and Approximation," ISNM, Vol. 20, p. 498, Birkhäuser, Basel, 1972.